

Low temperature spin glass fluctuations: expanding around a spherical approximation

A. Crisanti¹, C. De Dominicis², and T. Sarlat²

¹ Dipartimento di Fisica, Università di Roma *La Sapienza* and SMC, P.le Aldo Moro 2, I-00185 Roma, Italy.

² Institut de Physique Théorique, CEA - Saclay - Orme des Merisiers, 91191 Gif sur Yvette, France.

Received: V 7.5.4 2009/11/03 15:56:09 AC / Revised version: date

Abstract. The spin glass behavior near zero temperature is a complicated matter. To get an easier access to the spin glass order parameter $Q(x)$ and, at the same time, keep track of Q_{ab} , its matrix aspect, and hence of the Hessian controlling stability, we investigate an expansion of the replicated free energy functional around its “spherical” approximation. This expansion is obtained by introducing a constraint-field and a (double) Legendre Transform expressed in terms of spin correlators and constraint-field correlators. The spherical approximation has the spin fluctuations treated with a global constraint and the expansion of the Legendre Transformed functional brings them closer and closer to the Ising local constraint. In this paper we examine the first contribution of the systematic corrections to the spherical starting point.

PACS. 75.10.Nr Spin-glass and other random models – 64.70.Pf Glass transitions

1 Introduction

The infinite range Sherrington-Kirkpatrick (SK) model is defined by [1]:

$$\mathcal{H}_{\text{SK}} = -\frac{1}{2} \sum_{ij}^{1,N} J_{ij} \sigma_i \sigma_j \quad (1)$$

where the σ_i 's are ± 1 Ising spins and the couplings J_{ij} independent Gaussian random variables with null mean and variance equal to $1/N$. This model is solved with the

replica trick, that is all the thermodynamical information is encoded in the $n \rightarrow 0$ limit of the disorder-averaged replicated partition function:

$$\overline{Z^n} \equiv \overline{\exp \left(-\beta \sum_{a=1}^n \mathcal{H}_{\text{SK}}(\boldsymbol{\sigma}_a) \right)} \quad (2)$$

where $\beta = 1/T$ is the inverse of temperature and, as usual, $\overline{(\dots)}$ the average over the quenched disorder. Performing this average, and introducing the auxiliary symmetric replica matrix $Q_{ab} \equiv \frac{1}{N} \sum_{i=1}^N \sigma_{ia} \sigma_{ib}$ with $a \neq b$, leads to

[1]:

$$\overline{Z^n} = \int \prod_{ab} \sqrt{\frac{N\beta^2}{2\pi}} dQ_{ab} e^{N\mathcal{L}[Q]} \quad (3)$$

with the effective Lagrangian (density) [2]:

$$\mathcal{L}[Q] = -\frac{\beta^2}{4} \sum_{ab} Q_{ab}^2 + \Omega[Q] \quad (4)$$

$$\Omega[Q] = \ln \text{Tr}_{\sigma_a} \exp \left(\frac{\beta^2}{2} \sum_{ab} Q_{ab} \sigma_a \sigma_b \right). \quad (5)$$

The normalization factor in eq. (3) gives a sub-leading contribution for $N \rightarrow \infty$ and is omitted in the following.

In the thermodynamical limit, $N \rightarrow \infty$, the value of the integral in (3) is given by the saddle-point value $-n\beta f_{\text{SK}} = \mathcal{L}[Q]$, where Q_{ab} is evaluated from the stationarity condition:

$$\frac{\delta}{\delta Q_{ab}} \mathcal{L}[Q] = 0, \quad a \neq b \quad (6)$$

that is,

$$Q_{ab} = \langle \sigma_a \sigma_b \rangle. \quad (7)$$

The average is taken with respect to the weight

$$\zeta(\sigma) = e^{\frac{\beta^2}{2} \sum_{ab} Q_{ab} \sigma_a \sigma_b} / \text{Tr}_{\sigma} e^{\frac{\beta^2}{2} \sum_{ab} Q_{ab} \sigma_a \sigma_b}. \quad (8)$$

If instead, one had in mind to write the effective Lagrangian governing a short-range (*e.g.* nearest-neighbor) system, one would consider:

$$\begin{aligned} e^{\mathcal{W}[H]} &= \int \prod_{i,ab} dQ_{ab}(i) \exp \left(-\frac{\beta^2}{4} \sum_p \sum_{ab} (p^2 + 1) Q_{ab}^2(p) \right. \\ &\quad \left. + \sum_i \Omega[Q(i)] + \frac{1}{2} \sum_{i,ab} H_{ab}(i) Q_{ab}(i) \right). \end{aligned} \quad (9)$$

Here $Q_{ab}(p)$ is the (space) Fourier Transform of the site-dependent replica overlap matrix $Q_{ab}(i)$, with i the site-index, $H_{ab}(i)$ is an external (unphysical) field that couples to $Q_{ab}(i)$, and $\Omega[Q(i)]$ is given for each site i by (5).

Properties of $\mathcal{W}[H]$, and of its Legendre Transform $\Gamma[Q]$,

defined *via*:

$$\mathcal{W}[H] + \Gamma[Q] = \frac{1}{2} \sum_{i,ab} H_{ab}(i) Q_{ab}(i) \quad (10)$$

and,

$$\frac{\delta}{\delta Q_{ab}(i)} \Gamma[Q] = H_{ab}(i) \quad (11)$$

are obtained by writing:

$$Q_{ab}(i) = Q_{ab} + \delta Q_{ab}(i) \quad (12)$$

with Q_{ab} given as the (site-independent) saddle-point value of \mathcal{L} , eq. (6). An expansion of (9) in powers of $\delta Q_{ab}(i)$ shows that no linear terms (*i.e.* tadpoles) remains by virtue of (6). The quadratic term is given by:

$$\frac{1}{8T^2} \sum_p \sum_{abcd} \delta Q_{ab}(p) \mathcal{M}^{ab,cd}(p) \delta Q_{cd}(p) \quad (13)$$

where \mathcal{M} is the Hessian matrix whose eigenvalues are the so-called bare masses of the δQ correlation functions (*i.e.* the inverse propagator). With these notations, one has:

$$\mathcal{M}^{ab,cd}(p) = (p^2 + 1) \delta_{ab,cd}^{kr} - T^2 \frac{\delta^2}{\delta Q_{ab} \delta Q_{cd}} \Omega[Q]. \quad (14)$$

The higher order terms are in turn the couplings (cubic, quartic,...) of the fields δQ .

Whatever the adopted viewpoint, one cannot avoid the difficult construction of the Ω functional as given in (5). Parisi and others [5,6,7] have shown how to obtain solutions with R steps of Replica Symmetry Breaking (RSB) and in particular with $R \rightarrow \infty$, and how to construct equations satisfied by $Q(x)$, the continuous limit of the order parameter Q_{ab} for $R \rightarrow \infty$ [8]. These equations can be solved in the full low temperature phase [9]. Clearly,

however, for very low temperatures or for null temperature, the problem is delicate, as one can experience when expanding $\zeta(\sigma)$ in powers of $1/T^2$ [see eq. (8)]. The properties of the Parisi solution $Q(x)$ at very low temperatures have attracted some work in last years. The analysis has been carried on either directly on the continuous limit, i.e. taking first $R \rightarrow \infty$ and then $T \ll 1$ [10,11], or on the $T \rightarrow 0$ limit of large R -step replica symmetry breaking solutions [12,13,14,15]. In both approaches, however, properties of the Hessian remain elusive since once one is working with $Q(x)$ the matrix structure of the order parameter Q_{ab} is lost. On the other hand taking first $T \rightarrow 0$ and then $R \gg 1$ one encounters problems following the delicate commutativity of the two limits, extensively discussed in Ref. [16]. These two faced difficulties on the road to obtaining the Hessian (and eigenvalues) have motivated this work. The starting point is the observation that, if the strict “local” constraint $\sigma_a^2 = 1$ is replaced, as a first step, by the global constraint $\sum_a \sigma_a^2 = n$ (a kind of *spherical approximation* in replica space) then, the temperature behavior gets simpler while keeping the matrix structure of the problem. We show below how to systematically return towards the local constraint introducing higher and higher correlations generated under a double Legendre Transform on the Ω functional. In the resulting approximated functional for Ω , the null temperature limit becomes non-singular, and, at the same time, one keeps the matrix structure of Q_{ab} , leaving access to the Hessian.

The outline of the paper is as follows: in Section 2 we describe the general formalism based on the double Leg-

endre Transform and the stability of the saddle-point. In Sections 3 and 4 we discuss respectively the (trivial) zeroth order approximation and the first non-trivial approximation, the two-replica approximation, of the functional Ω . To make the paper also accessible to uninitiated readers, we have deferred to appendices, besides a couple of detailed calculations, a reminder concerning some of the technical tools used in the text (Replica Fourier Transform and double Legendre Transform).

2 General formalism

2.1 Equation of motion:

For spherical models, the spin variables σ_a are continuous: the trace over the spins becomes an n -dimensional Gaussian integral. Here, the Ising spins are discrete variables $\sigma_a = \pm 1$. To overcome the discrete nature of the spin, one introduces a constraint-field:

$$\begin{aligned} \text{Tr}_{\sigma}(\dots) &= \prod_{a=1}^n \sum_{\sigma_a=\pm 1} (\dots) \\ &= \prod_{a=1}^n \int_{-\infty}^{+\infty} d\sigma_a [\delta(\sigma_a + 1) + \delta(\sigma_a - 1)] (\dots) \\ &= \int_{-\infty}^{+\infty} \prod_{a=1}^n d\sigma_a \int_{-i\infty}^{+i\infty} \prod_{a=1}^n \frac{d\lambda_a}{2\pi i} \\ &\quad \times \exp \left[-\frac{1}{2} \sum_a \lambda_a (\sigma_a^2 - 1) \right] (\dots). \end{aligned} \quad (15)$$

The auxiliary variables λ_a control the fluctuations of the continuous spin variables σ_a around the Ising values ± 1 and thus we expect that they diverge as $T \rightarrow 0$.

By inserting (15) into the expression (5), the functional $\mathcal{L}[Q]$ can be written as:

$$\mathcal{L}[Q] = -\frac{\beta^2}{4} \sum_{ab}^{1,n} Q_{ab}^2 + \Omega_{0+}[Q] - \frac{n\beta^2}{4} \quad (16)$$

where

$$\begin{aligned}\Omega_{0+}[Q] &= \lim_{\epsilon \rightarrow 0^+} \Omega_\epsilon[Q] \\ &= \lim_{\epsilon \rightarrow 0^+} \ln \int_{-\infty}^{+\infty} d\sigma \int_{-i\infty}^{+i\infty} \frac{d\lambda}{(2\pi i)^n} e^{-S[\sigma, \lambda]} \quad (17)\end{aligned}$$

and $S[\sigma, \lambda]$ is given by,

$$\begin{aligned}S[\sigma, \lambda] &= -\frac{\beta^2}{2} \sum_{ab} Q_{ab} \sigma_a \sigma_b - \frac{\epsilon}{2} \sum_a \lambda_a^2 \\ &\quad + \frac{1}{2} \sum_a \lambda_a \sigma_a^2 - \frac{1}{2} \sum_a \lambda_a . \quad (18)\end{aligned}$$

The constant term $-n\beta^2/4$ follows from the (definition of the) diagonal terms $Q_{aa} = 1$ of the overlap matrix. We recall that when introduced, the overlap matrix is defined only for $a \neq b$, the reason being that for Ising spins $\sigma^2 = 1$. The diagonal terms Q_{aa} are then usually (tacitly) set to zero, as in the previous Sections. This choice may not be the most suitable for spherical models where sums over spin values are replaced by Gaussian integrals. We thus (re)define the overlap matrix with the diagonal terms equal to 1. The constant term comes out then for consistency with eqs. (4) and (5).

The functional $\Omega_\epsilon[Q]$ can be expressed in terms of the field expectation values and two-point correlation functions by introducing linear and bilinear auxiliary fields coupled with σ and λ , and performing a double Legendre Transform with respect to these fields [17,18,19]. See also Appendix D. In absence of an external field coupled to σ_a the average $\langle \sigma_a \rangle$ vanishes. Moreover, one can check that $\langle \sigma_a \lambda_b \rangle = 0$ [20]. Therefore the relevant field averages and correlation functions are [21]

$$\langle \lambda_a \rangle = \lambda \quad (19)$$

and

$$\langle \sigma_a \sigma_b \rangle = G_{ab}, \quad \langle \lambda_a \lambda_b \rangle = \Lambda_{ab} . \quad (20)$$

All averages are evaluated with the weight $\exp(-S[\sigma, \lambda])$.

After the double Legendre Transform has been taken, and the auxiliary linear and bilinear fields set to zero, the functional $\Omega_\epsilon[Q]$ comes out to be [18]:

$$\begin{aligned}\Omega_\epsilon[Q] &= -S[\mathbf{0}, \lambda] - \frac{1}{2} \text{Tr} \ln G^{-1} - \frac{1}{2} \text{Tr} \ln \Lambda^{-1} \\ &\quad - \frac{1}{2} \text{Tr} D^{-1} G - \frac{1}{2} \text{Tr} \Delta^{-1} \Lambda - \mathcal{K}_2[G, \Lambda] + n \quad (21)\end{aligned}$$

where,

$$(D^{-1})_{ab} = \lambda \delta_{ab} - \beta^2 Q_{ab}, \quad (\Delta^{-1})_{ab} = -\epsilon \delta_{ab} . \quad (22)$$

Here, $\mathcal{K}_2[G, \Lambda]$ is given by the sum of all the two-particle irreducible (2-PI) vacuum graphs in a theory with vertices

$$-\frac{1}{2} \lambda_a \sigma_a^2 = \text{~~~~~} \bullet \text{~~~} \quad (23)$$

and propagators

$$G_{ab} = \text{~~~~~} a \text{~~~~~} b , \quad \Lambda_{ab} = \text{~~~~~} a \text{~~~~~} b . \quad (24)$$

The first terms of the diagrammatic expansion of $\mathcal{K}_2[G, \Lambda]$ are:

$$\begin{aligned}\mathcal{K}_2[G, \Gamma] &= \text{~~~~~} a \text{~~~~~} b \\ &\quad + \text{~~~~~} a \text{~~~~~} c \\ &\quad + \cdots . \quad (25)\end{aligned}$$

By construction, when the auxiliary fields are set to zero the double Legendre Transform is stationary with respect to the variations of λ , G and Λ . This leads respectively to:

$$G_{aa} = 1 \quad (26)$$

$$(G^{-1})_{ab} = (D^{-1})_{ab} + 2 \frac{\partial}{\partial G_{ab}} \mathcal{K}_2[G, \Lambda] \quad (27)$$

$$(\Lambda^{-1})_{ab} = 2 \frac{\partial}{\partial \Lambda_{ab}} \mathcal{K}_2[G, \Lambda] . \quad (28)$$

Equations (27) and (28) are known as the Dyson equation for the two-point functions G and Λ with self-energies:

$$\Sigma_{ab}^{\sigma\sigma} = -2 \frac{\partial}{\partial G_{ab}} \mathcal{K}_2[G, \Lambda] \quad (29)$$

$$\Sigma_{ab}^{\lambda\lambda} = -2 \frac{\partial}{\partial \Lambda_{ab}} \mathcal{K}_2[G, \Lambda] . \quad (30)$$

These Dyson equations result from the Legendre Transform, and give *exact* relations for G and Λ . By solving them, the propagators G_{ab} and Λ_{ab} become functionals of Q_{ab} that, substituted back into (21), give $\Omega[Q]$ as functional of Q_{ab} alone. By using now the saddle-point equation (6) one can write the self-consistent equation for Q_{ab} , that reads

$$Q_{ab} = G_{ab}, \quad a \neq b . \quad (31)$$

With the choice $Q_{aa} = 1$ eqs. (26) and (31) can be merged into the single equation of motion

$$Q_{ab} = G_{ab} \quad \forall a, b . \quad (32)$$

We stress that, while in the following we use the stationarity conditions in this form, one should keep in mind that the diagonal and off-diagonal terms of these equations follow from two distinct conditions. The diagonal terms follow from the double Legendre Transform, and our choice $Q_{aa} = 1$. This is an exact relation. The off-diagonal terms follow from the saddle point eq. (6) and hence eq. (31) is valid only in the thermodynamic limit.

2.2 Stability

To decide whether the solutions of the equation of motion are stable or not, one has to expand the effective Lagrangian to second order in δQ :

$$\begin{aligned} \mathcal{L}\{Q_{ab} + \delta Q_{ab}(p)\} &= \mathcal{L}^{(0)}\{Q_{ab}\} \\ &+ \frac{\beta^2}{2} \sum_{ab} \delta Q_{ab}(0)(G_{ab} - Q_{ab}) \\ &- \frac{\beta^2}{8} \sum_p \sum_{abcd} \delta Q_{ab}(p) \mathcal{M}^{ab,cd}(p) \delta Q_{cd}(p) \end{aligned} \quad (33)$$

with the Hessian matrix:

$$\mathcal{M}^{ab,cd}(p) = (p^2 + 1) \delta_{ab,cd}^{\text{Kr}} - \frac{\delta G_{ab}}{\delta Q_{cd}} . \quad (34)$$

Let us introduce:

$$\mathcal{X}^{ab,cd} \equiv \frac{\delta G_{ab}}{\delta Q_{cd}} \quad (35)$$

$$\mathcal{Y}^{ab,cd} \equiv \frac{\delta \Lambda_{ab}}{\delta Q_{cd}} \quad (36)$$

which through the equations of motion (27) and (28) satisfy:¹

$$\begin{aligned} &- \sum_{ef} \left[G_{ae}^{-1} G_{bf}^{-1} + G_{af}^{-1} G_{be}^{-1} + \frac{\delta^2}{\delta G_{ab} \delta G_{ef}} \mathcal{K}_2 \right] \mathcal{X}^{ef,cd} \\ &- \sum_{ef} \left(\frac{\delta^2}{\delta G_{ab} \delta \Lambda_{ef}} \mathcal{K}_2 \right) \mathcal{Y}^{ef,cd} = -\frac{\delta_{ab,cd}^{\text{Kr}}}{T^2} \end{aligned} \quad (37)$$

$$\begin{aligned} &- \sum_{ef} \left[\Lambda_{ae}^{-1} \Lambda_{bf}^{-1} + \Lambda_{af}^{-1} \Lambda_{be}^{-1} + \frac{\delta^2}{\delta \Lambda_{ab} \delta \Lambda_{ef}} \mathcal{K}_2 \right] \mathcal{Y}^{ef,cd} \\ &- \sum_{ef} \left(\frac{\delta^2}{\delta \Lambda_{ab} \delta G_{ef}} \mathcal{K}_2 \right) \mathcal{X}^{ef,cd} = 0 . \end{aligned} \quad (38)$$

Solving (38) for \mathcal{Y} , and inserting it into (37), one has:

$$\begin{aligned} &- \sum_{ef} \left[G_{ae}^{-1} G_{bf}^{-1} + G_{af}^{-1} G_{be}^{-1} + \frac{\delta^2}{\delta G_{ab} \delta G_{ef}} \mathcal{K}_2 \right. \\ &\quad \left. - \sum_{gh} \frac{\delta^2}{\delta G_{ab} \delta \Lambda_{gh}} \mathcal{K}_2 \sum_{ij} \mathcal{D}_{gh,ij}^{-1} \frac{\delta^2}{\delta \Lambda_{ij} \delta G_{ef}} \mathcal{K}_2 \right] \mathcal{X}^{ef,cd} \\ &= -\frac{\delta_{ab,cd}^{\text{Kr}}}{T^2} . \end{aligned} \quad (39)$$

¹ Please note that in eqs. (37)-(39) we used G_{ab}^{-1} for $(G^{-1})_{ab}$ to bypass cumbersome writing.

The matrix \mathcal{D} is the one multiplying \mathcal{Y} in (38). At this point, we can now use the equality $G_{ab} = Q_{ab}$, resulting from the stationarity condition.

To solve the self-consistent equations of motion, we have to specify the form of the matrix Q_{ab} that takes into account possible breaking of the permutation symmetry of replica pairs. A standard parameterization has been introduced by Parisi [5,6]. The overlap matrix Q_{ab} for R allowed breaking in the replica permutation symmetry consists in dividing Q_{ab} into successive boxes of decreasing size p_r , with $p_0 = n$ and $p_{R+1} = 1$, along the diagonal and assigning the elements Q_{ab} of the overlap matrix so that

$$Q_{ab} \equiv Q_{a \cap b = r} \equiv Q_r, \quad r = 0, \dots, R+1 \quad (40)$$

with $1 = Q_{R+1} \geq Q_R \geq \dots \geq Q_1 \geq Q_0$. The notation $a \cap b = r$ means that a and b belong to the same box of size p_r but to two *distinct* boxes of size $p_{r+1} < p_r$. The element Q_0 must vanish in absence of external fields that break the up/down symmetry [22]. In the following, if not explicitly stated, this will be always assumed.

In this formalism the Replicon component of the Hessian (that usually harbors the most dangerous, *i.e.*, the lowest eigenvalues), becomes [23,24]

$$\mathcal{M}_{Rep}^{ab,cd} = \mathcal{M}_{uv}^{rr} \quad (41)$$

where $a \cap b \equiv c \cap d = r$ is the common overlap while $u = \max\{a \cap c, a \cap d\}$ and $v = \max\{b \cap c, b \cap d\}$ the cross-overlaps. Replicon matrices enjoy the property of being diagonalized under a double Replica Fourier Transform (RTF), and matrix products of such matrices becoming or-

dinary products of RFT matrices (like in ordinary Fourier Transform), we thus get [24]:

$$\mathcal{M}_{\hat{k}\hat{l}}^{rr} = (p^2 + 1) - \mathcal{X}_{\hat{k}\hat{l}}^{rr} \quad (42)$$

$$\frac{1}{\mathcal{X}_{\hat{k}\hat{l}}^{rr}} = T^2 \left[\frac{1}{\widehat{G}_k \widehat{G}_l} + \mathcal{K}_{\hat{k}\hat{l}}^{G_r G_r} - \mathcal{K}_{\hat{k}\hat{l}}^{G_r \Lambda_r} \frac{1}{\mathcal{D}_{\hat{k}\hat{l}}^{rr}} \mathcal{K}_{\hat{k}\hat{l}}^{\Lambda_r G_r} \right] \quad (43)$$

$$\mathcal{D}_{\hat{k}\hat{l}}^{rr} = \frac{1}{\widehat{\Lambda}_k \widehat{\Lambda}_l} + \mathcal{K}_{\hat{k}\hat{l}}^{\Lambda_r \Lambda_r} \quad (44)$$

where $r = 0, \dots, R$ and $k, l = r+1, \dots, R+1$. By $\mathcal{K}_{\hat{k}\hat{l}}^{G_r G_r}$ we mean the double RFT on the Replicon component of $\frac{\delta^2}{\delta G_{ab} \delta G_{cd}} \mathcal{K}_2$ with respect the lower indices (cross overlaps) u and v . The “hat” denotes the RFT (see Appendix A).

3 Zero order approximation

The zeroth order approximation is obtained by neglecting the 2-PI contributions of $\mathcal{K}_2[G, \Lambda]$. Combining eqs. (22), (27) and (32) one is led to

$$(Q^{-1})_{ab} = \lambda \delta_{ab} - \beta^2 Q_{ab}. \quad (45)$$

To solve this equation the parameterization of the Q_{ab} matrix is needed. By using Parisi’s parameterization one gets for the off-diagonal terms

$$(Q^{-1})_r = -\beta^2 Q_r, \quad r = 0, \dots, R. \quad (46)$$

As shown in Appendix C, for any R , this equation only admits the solution [25]

$$Q_r = Q = 1 - T, \quad \forall r \quad (47)$$

i.e., the Replica Symmetric (RS) solution. The value of λ is set by the diagonal term $a = b$ of eq. (45) and reads

$$\lambda = \frac{2}{T}. \quad (48)$$

The stability of the solution is ruled by the Replicon eigenvalue [see eqs. (42)-(44)],

$$\mathcal{M}_{\hat{1}\hat{1}}^{00}(p=0) = 1 - \beta^2(1-Q)^2 = 0, \quad (49)$$

where $r = 0$ and the (Fourier transformed) cross-overlap

$k = l = 1$. The solution is then marginally stable, as known for the 2-spin spherical model.

Let us note a technical point. In this approximation $\Lambda_{ab} = -\epsilon\delta_{ab}$, so that the term $\text{Tr ln } \Lambda$ is singular as $\epsilon \rightarrow 0^+$. The functional $\mathcal{L}[Q]$ is nevertheless regular in this limit. The reason is that neglecting the $\mathcal{K}_2[G, \Lambda]$ contribution is equivalent to remove the Ising-like constraint on the spin variables. This is also equivalent to discard the last two terms of $S[\sigma, \lambda]$ in eq. (18). A proper normalization is now needed to regularize the $\epsilon \rightarrow 0^+$ limit of $\Omega_\epsilon[Q]$.

This introduces an additional constant term in $\mathcal{L}[Q]$ that exactly balances the singularity coming from $\text{Tr ln } \Lambda$.

4 Two-replica approximation

The two-replica approximation consists in taking only the first 2-PI diagram in the diagrammatic expansion of $\mathcal{K}_2[G, \Lambda]$, namely,

$$\begin{aligned} \mathcal{K}_2[G, \Lambda] &= \text{Diagram with a wavy line connecting } a \text{ and } b \text{ inside a circle} \\ &= \frac{1}{2} \left(-\frac{1}{2} \right)^2 2 \sum_{ab} G_{ab}^2 \Lambda_{ab}. \end{aligned} \quad (50)$$

The functional $\mathcal{L}[Q]$ then reads

$$\begin{aligned} \mathcal{L}[Q] &= \frac{\beta^2}{4} \sum_{ab} Q_{ab}^2 + \frac{1}{2} \text{Tr ln } G + \frac{1}{2} \text{Tr ln } \Lambda \\ &\quad - \frac{1}{4} \sum_{ab} G_{ab}^2 \Lambda_{ab} \end{aligned} \quad (51)$$

with the matrices G_{ab} and Λ_{ab} solution of the Dyson equations

$$(G^{-1})_{ab} = \lambda \delta_{ab} - \beta^2 Q_{ab} + G_{ab} \Lambda_{ab} \quad (52)$$

$$(\Lambda^{-1})_{ab} = \frac{1}{2} G_{ab}^2 \quad (53)$$

and $G_{aa} = 1$. Stationarity of $\mathcal{L}[Q]$ with respect to the variations of Q_{ab} leads to the self-consistent equation $Q_{ab} = G_{ab}$, the form of which is dictated by the *ansatz* used for the matrix Q_{ab} .

The Replicon component for the two-replica approximation reads:

$$\begin{aligned} \mathcal{M}_{\hat{k}\hat{l}}^{rr}(p) &= p^2 + 1 \\ &\quad - \beta^2 \left[\frac{1}{\widehat{Q}_k \widehat{Q}_l} + \Lambda_r - Q_r \widehat{\Lambda}_k \widehat{\Lambda}_l Q_r \right]^{-1} \end{aligned} \quad (54)$$

with $r = 0, \dots, R$ and $k, l = r+1, \dots, R+1$.

In the next subsection we examine the existence and stability of three possible scenarios, corresponding to three particular parameterizations of the Parisi matrix: the Replica Symmetric, the one step Replica Symmetry Breaking and the full Replica Symmetry Breaking ansatz.

In the Replica Symmetric solution all replicas are treated on an equal footing, and the matrices Q_{ab} and Λ_{ab} are given by

$$Q_{ab} = (1-Q) \delta_{ab} + Q, \quad \Lambda_{ab} = (\Lambda_1 - \Lambda_0) \delta_{ab} + \Lambda_0. \quad (55)$$

Then from eq. (53) and the relation $Q_{ab} = G_{ab}$ we have for $n \rightarrow 0$

$$\Lambda_1 - \Lambda_0 = \frac{2}{1-Q^2}, \quad \Lambda_0 = -\frac{2Q^2}{(1-Q^2)^2} \quad (56)$$

that, inserted back into (52), leads to the RS equation

$$\frac{1-2Q}{(1-Q)^2} = \lambda - \beta^2 + 2\frac{1-2Q^2}{(1-Q^2)^2} \quad (57)$$

$$\frac{Q}{(1-Q)^2} = \beta^2 Q + 2\frac{Q^3}{(1-Q^2)^2}. \quad (58)$$

These equations can be easily solved for λ and β as function of $Q \in [0, 1]$, and one ends up with

$$\lambda = \frac{2Q}{1-Q^2} \quad (59)$$

$$\beta^2 = \frac{1+2Q-Q^2}{(1-Q^2)^2}. \quad (60)$$

By varying Q between 0 and 1 one obtains the solution, if it exists, in the whole range of physical values of Q . We note in particular that Q vanishes as $T \rightarrow 1^-$, while $Q \rightarrow 1$ as T goes to 0:

$$Q = 1 - \frac{T}{\sqrt{2}} + O(T^2), \quad T \rightarrow 0. \quad (61)$$

The stability of the RS solution is ruled by the Replicon eigenvalue

$$\begin{aligned} \mathcal{M}_{11}^{00}(p=0) &= 1 - \frac{\beta^2}{(1-Q)^{-2} + \Lambda_0 - Q^2(\Lambda_1 - \Lambda_0)^2} \\ &= \frac{4Q^2}{5Q^2 - 2Q - 1}. \end{aligned} \quad (62)$$

The Replicon eigenvalue is positive at $T = 0$, where its value is 2, and remains positive up to the temperature $T^* \simeq 0.34\dots$, where the denominator vanishes. For larger temperatures the Replicon is negative, and vanishes as $T \rightarrow 1^-$. Therefore we conclude that the RS solutions exists for all temperatures below $T = 1$, as in the trivial case. But this solution is only stable for $T < T^*$. This is a rather unusual feature of this approximation.

4.2 The one step Replica Symmetry Breaking (1RSB) solution

In the previous section we have seen that the first correction coming from $\mathcal{K}_2[G, \Lambda]$ is not enough to completely destabilize the RS solution that exists for $T < 1$, yet its domain of validity is reduced to the temperature range up to 0.34.... In this section we investigate the existence of solutions parameterized by the so-called one step RSB ansatz:

$$Q_{ab} = (Q_2 - Q_1) \delta_{ab} + (Q_1 - Q_0) \epsilon_{ab} + Q_0 \quad (63)$$

where the matrix ϵ_{ab} is equal to 1 if a and b are in anyone of the diagonal boxes of size $p_1 \equiv m$, and 0 otherwise. The diagonal element $Q_{R+1} = 1$ of the overlap matrix writes $Q_2 = 1$. The matrices G_{ab} and Λ_{ab} have a similar structure.

In the absence of an external field that breaks the up/down symmetry, $Q_0 = 0$. This in turn implies that all 0-indexed quantities, such as Λ_0 and G_0 , must also vanish. It is not difficult to show that $Q_0 = \Lambda_0 = G_0 = 0$ is a solution of the self-consistent equations of motion.

Inserting the 1RSB parameterization into eq. (53), using the condition $G_{ab} = Q_{ab}$, and taking the $n \rightarrow 0$ limit, we have

$$\Lambda_2 - \Lambda_1 = \frac{2}{1-Q_1^2}, \quad (64)$$

$$\Lambda_1 = -\frac{2Q_1^2}{(1-Q_1^2)(1-Q_1^2 + mQ_1^2)}. \quad (65)$$

Note that for $m \rightarrow 0$ these relations reduce to those found for the RS case. This is not unexpected since if $Q_0 = 0$ the systems breaks down into n/m subsystems of dimension

m , each one having a RS structure, Q_1 playing the role of Q . For $m \rightarrow 0$ we recover the RS solution.

The other Dyson equation, as of eq. (52), leads to

$$\lambda = \frac{(2-m)Q_1}{(1+Q_1)(1-Q_1+mQ_1)} \quad (66)$$

$$\beta^2 = \frac{1+2Q_1+(m-1)Q_1^2}{(1+Q_1)(1+Q_1-mQ_1)(1+Q_1^2-mQ_1^2)}. \quad (67)$$

As for the RS solution by varying Q_1 in the range $[0, 1]$ we get, for any value of $m \in [0, 1]$ a 1RSB solution, if it exists.

To fix the value of m we use the stationarity equation $(\partial/\partial m)\mathcal{L}[Q] = 0$ (yielding the so-called *static solution* [26,27,28,29]). The computation is achieved using the formulae of Appendices A and B, leading to

$$\begin{aligned} & \frac{\beta^2}{2}Q_1^2 + \frac{1}{m^2} \ln \left[\frac{1-Q_1}{1-Q_1+mQ_1} \right] \\ & + \frac{1}{m^2} \ln \left[\frac{\Lambda_2-\Lambda_1}{\Lambda_2-\Lambda_1+m\Lambda_1} \right] \\ & + \frac{1}{m} \frac{Q_1}{1-Q_1+mQ_1} + \frac{1}{m} \frac{\Lambda_1}{\Lambda_2-\Lambda_1+m\Lambda_1} \\ & - \frac{1}{2}Q_1^2\Lambda_1 = 0. \end{aligned} \quad (68)$$

By eliminating Λ_1 , Λ_2 and β in favor of Q_1 and m with the help of eqs. (64)-(65) and (67), we end up with the following equation

$$\begin{aligned} & 2(1-Q_1+mQ_1) \\ & \times (1+Q_1) \ln \left[\frac{1-Q_1^2+mQ_1^2}{(1+Q_1)(1-Q_1+mQ_1)} \right] \\ & + mQ_1(2+mQ_1) = 0 \end{aligned} \quad (69)$$

that gives $m \equiv m(Q_1)$ for the 1RSB case. Solving this equation shows that $m(Q_1)$ is a monotonically increasing function of Q_1 that vanishes for $Q_1 = 0$, i.e., $T = 1$, while $\lim_{Q_1 \rightarrow 1} m(Q_1) = 4 \ln 2 - 2 < 1$.

It turns out that the 1RSB solution appears at the critical temperature $T = 1$ and exists down to the lower critical temperature $T_{1\text{RSB}} \simeq 0.656485\dots$ where Q_1 reaches the maximum allowed value $Q_1 = 1$.

The stability of the 1RSB solution is now ruled by two Replicon eigenvalues,

$$\mathcal{M}_{11}^{00}(p=0) = 1 - \beta^2(1-Q_1)^2 \quad (70)$$

and

$$\begin{aligned} \mathcal{M}_{22}^{11}(p=0) &= 1 \\ & - \frac{\beta^2}{(1-Q_1)^{-2} + \Lambda_1 - Q_1^2(\Lambda_2 - \Lambda_1)^{-2}}. \end{aligned} \quad (71)$$

At $T = 1$ both eigenvalues vanish. These two eigenvalues cross each other at $T_0 = 0.6760$. Above T_0 , \mathcal{M}_{22}^{11} is the lowest eigenvalues and is negative. Below T_0 , \mathcal{M}_{11}^{00} is also negative. Therefore the solution is always unstable.

4.3 The full Replica Symmetry Breaking (full-RSB) solution

We consider now the solution, if any, with an infinite number of replica symmetry breaking steps (full-RSB). In this limit the matrices Q_{ab} , G_{ab} and Λ_{ab} are described by functions of a single parameter x varying between 0 and 1. To find the self-consistent equation of motion of the full-RSB solution let us consider the case of R replica symmetry breaking steps. The full-RSB solution is obtained as the limit $R \rightarrow \infty$.

By assuming a Parisi's R replica symmetry breaking structure for the matrices Q_{ab} , G_{ab} and Λ_{ab} , the Dyson equations (52) and (53) for the non-diagonal terms be-

comes

$$(G^{-1})_r = -\beta^2 Q_r + Q_r \Lambda_r , \quad (72)$$

$$(\Lambda^{-1})_r = \frac{1}{2} Q_r^2 , \quad r = 0, \dots, R . \quad (73)$$

These equations are solved through the Replica Fourier Transform (RFT) (see Appendix A for proper definition and Appendix B for an example). The RFT of eq. (73) reads

$$\widehat{\Lambda^{-1}}_k = \frac{1}{\widehat{\Lambda}_k} = \frac{1}{2} \widehat{q}_k \quad (74)$$

where we have introduced the shorthand $q_{ab} = (Q_{ab})^2$.

Inverting the RFT yields in the continuous limit $R \rightarrow \infty$

$$\begin{aligned} \Lambda(x) &= - \int_0^x ds \frac{1}{s} \frac{d}{ds} \widehat{\Lambda}(s) \\ &= 2 \int_0^x ds \frac{1}{\widehat{q}(s)^2} \frac{d}{ds} \widehat{q}(s) \end{aligned} \quad (75)$$

where,

$$\widehat{q}(x) = 1 - Q(x_c)^2 + \int_x^{x_c} ds s \frac{d}{ds} Q(s)^2 \quad (76)$$

and

$$x_c = \lim_{R \rightarrow \infty} p_R , \quad Q(x_c) = \lim_{R \rightarrow \infty} Q_R . \quad (77)$$

Finally, from eqs. (72), (75) and (76) we have the equation

of motion for $Q(x)$ for $0 \leq x \leq x_c$:

$$\int_0^x ds \frac{\dot{Q}(s)}{\widehat{Q}(s)^2} = \beta^2 Q(x) + 4 Q(x) \int_0^x ds \frac{Q(s) \dot{Q}(s)}{\widehat{q}(s)^2} \quad (78)$$

where the “dot” denotes the derivative of the function with respect to its argument. As done for the 1RSB solution we assumed that all 0-indexed quantities, such as $Q_0 \equiv Q(0)$ etc., vanish.

The solution of this integral equation is obtained by successive differentiation with respect to x . The first yields

$$\frac{1}{\widehat{Q}(x)^2} = \beta^2 + 4 \int_0^x ds \frac{Q(s) \dot{Q}(s)}{\widehat{q}(s)^2} + 4 \frac{Q(x)^2}{\widehat{q}(x)^2} . \quad (79)$$

By setting $x = 0$ we get

$$\widehat{Q}(0) = 1 - Q(x_c) + \int_0^{x_c} ds s \dot{Q}(s) = T . \quad (80)$$

As $Q(x) = Q(x_c)$ for $x \geq x_c$, equation (80) reads

$$1 - \int_0^1 dx Q(x) = T . \quad (81)$$

Notice that the same manipulations applied to the internal energy (per site) u , yield

$$\frac{1}{2} \left(1 - \int_0^1 dx Q^2(x) \right) = -uT . \quad (82)$$

The sum rules (81) and (82) are known to be exact for the SK model. They are verified here to the first order of the 2-PI expansion of \mathcal{K}_2 .

To solve the equation of motion (79) we take one more x derivative to get

$$x = 6 Q(x) \frac{\widehat{Q}(x)^3 \widehat{q}(x)}{\widehat{q}(x)^3 - 8 Q(x)^3 \widehat{Q}(x)^3} . \quad (83)$$

Taking advantage of

$$\begin{aligned} \widehat{Q}(Q) &= 1 - Q_c + \int_Q^{Q_c} dQ' x(Q') \\ &= 1 - Q_c - v(Q) \end{aligned} \quad (84)$$

$$\begin{aligned} \widehat{q}(Q) &= 1 - Q_c^2 + 2 \int_Q^{Q_c} dQ' Q' x(Q') \\ &= 1 - Q_c^2 - 2Q v(Q) + 2 r(Q) \end{aligned} \quad (85)$$

where $Q_c = Q(x_c)$ is the *plateau* value, it is easy to check that eqs. (83), (84) and (85) are equivalent to the coupled differential equations

$$\begin{aligned} \frac{d}{dQ} r(Q) &= v(Q) \\ \frac{d}{dQ} v(Q) &= x(Q) \end{aligned} \quad (86)$$

with the boundary condition

$$r(Q_c) = v(Q_c) = 0 . \quad (87)$$

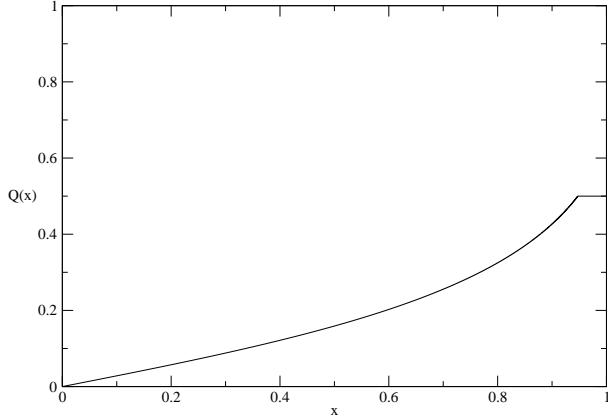


Fig. 1. $Q(x)$ versus x for the full-RSB solution. The solution is obtained with $Q_c = 0.5$ and corresponds to temperature $T = 0.8076\dots$

Solving these equations for $0 \leq Q_c \leq 1$, and fixing the temperature through eq. (80), we obtain the complete solution $Q(x)$. In Fig. 1 we show the form of $Q(x)$. As found for the 1RSB solution, the temperature does not vanishes as $Q_c \rightarrow 1$, but reaches the finite value $\lim_{Q_c \rightarrow 1} T(Q_c) = T_{\text{full-RSB}} = 0.7829\dots$

At variance with the 1RSB solution the continuous solution is marginally stable since the lowest Replicon eigenvalues vanishes. Indeed, looking at stability, we have for the Replicon eigenvalues:

$$\begin{aligned} \mathcal{M}_{\hat{k}\hat{l}}^{xx}(p) &= p^2 + 1 \\ &- \beta^2 \left[\frac{1}{\hat{Q}_k \hat{Q}_l} + \left(\frac{2}{q} \right)_x - 4Q_x^2 \frac{1}{\hat{q}_k \hat{q}_l} \right]^{-1}. \end{aligned} \quad (88)$$

Before letting $R \rightarrow \infty$ the lowest eigenvalue is $\mathcal{M}_{r+1r+1}^{rr}$. In the continuum \mathcal{M}_{x+x+}^{xx} is correspondingly given by,

$$\begin{aligned} \mathcal{M}_{x+x+}^{xx}(p) &= p^2 + 1 \\ &- \beta^2 \left[\left(\frac{1}{\hat{Q}_x} \right)^2 + \left(\frac{2}{q} \right)_x - 4Q_x^2 \left(\frac{1}{\hat{q}_x} \right)^2 \right]^{-1}. \end{aligned} \quad (89)$$

On (89), one identifies, from the equation of motion (79), that the quantity in the square bracket is equal to T^2 ,

leading to a zero mode:

$$\mathcal{M}_{x+x+}^{xx}(p) = p^2. \quad (90)$$

This is a check of the existence of Goldstone zero modes, accompanying the breakdown of a continuous invariance group, *i.e.* the existence of Ward-Takahashi identities [30]. From (88) one can also obtain the size of the band on top of the zero-modes \mathcal{M}_{x+x+}^{xx} by evaluating:

$$\begin{aligned} \mathcal{M}_{x_c x_c}^{00} - \mathcal{M}_{0+0+}^{00} &= \mathcal{M}_{x_c x_c}^{xx} \\ &= 1 - \frac{1}{T^2} \frac{1}{\left(\frac{1}{1-Q_1} \right)^2} \\ &= 1 - \beta^2 (1 - Q_1)^2. \end{aligned} \quad (91)$$

As a result at this level of approximation, one keeps the same structure as for the SK model near T_c : a Replicon broad band of modes, bordered below by zero modes.

5 Conclusion

In this work, we have introduced, for disordered Ising-like systems, an expansion around their spherical approximation in replica space via the use of constraint-fields and a double Legendre Transform of the free energy functional. Successive approximations are given by 2-PI graphs constructed with replicated spin correlators and replicated constraints correlators. A striking feature of this approach is that one can keep track of the matrix structure of the order parameter (and thus of the Hessian) at all temperatures.

We have applied the method to the SK model and analyzed in detail the lowest nontrivial contribution, studying

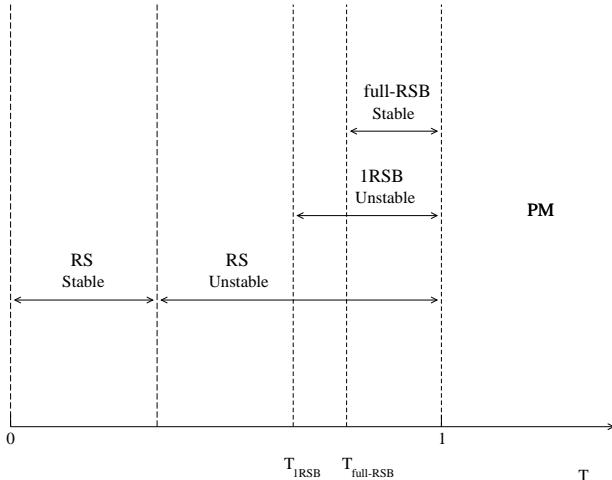


Fig. 2. Phase diagram in the two-replica approximation. The RS solution ($R = 0$) exists for all temperature below $T = 1$, but it is stable below $T^* = 0.34\dots$ only. The 1RSB solution ($R = 1$) exists in the temperature range $T_{\text{1RSB}} = 0.656\dots \leq T < 1$, but it is unstable. Finally the full-RSB solution ($R \rightarrow \infty$) exists in the temperature range $T_{\text{full-RSB}} \leq T < 1$, $T_{\text{full-RSB}} = 0.783\dots$, and it is marginally stable. Above $T = 1$ only the paramagnetic (PM) solution with $Q_{ab} = \delta_{ab}$ exists. All solutions coincide at $T = 1$.

in each case (RS, 1RSB, full-RSB) successively, the existence of solutions and their stability as summarized in Fig. 2. At high enough T , the constraint-field fluctuations are sufficient to destabilize the RS solution giving rise, instead, to a marginally stable continuous solution. At low enough T the constraint-field becomes ineffective: a stable RS solution is found.

The ineffectiveness of the constraint-field seems linked to the fact that, at its lowest order the 2-PI functional is linear in the constraint-field. This leads actually to a trivial relationship between the constraint-field and the order parameter and a correction to the free energy that is

linear in temperature, a very peculiar feature of the lowest order approximation. This remark motivates future work to take into account more generic features of the model as in the four replica approximation.

One of us (C.D.) would like to thank E. Brezin and H. Orland for useful discussions. A.C. would like to thank the IPhT of CEA, where part of this work was done, for hospitality and support.

A Inversion of a Parisi Matrix with the RFT formalism

In the R steps RSB scheme an $n \times n$ symmetric matrix Q_{ab} is parameterized by two sets of numbers. The first set $(p_0, p_1, \dots, p_R, p_{R+1})$, with $p_0 = n$ and $p_{R+1} = 1$, gives the size of the boxes the matrix is successively divided into. By definition for any finite n the p_r are decreasing integer numbers $p_r > p_{r+1}$, however in the $n \rightarrow 0$ limit they becomes real numbers in the range $[0, 1]$ and its order gets reversed, that is, $p_{r+1} > p_r$. The second set $(Q_0, Q_1, \dots, Q_R, Q_{R+1})$, gives the value of Q_{ab} in each box,

$$Q_{ab} \equiv Q_{a \cap b = r} = Q_r, \quad r = 0, \dots, R + 1. \quad (92)$$

The notation $a \cap b = r$ means that the indexes a and b belong to the same box of size p_r but to two *distinct* boxes of size p_{r+1} . The term Q_{R+1} is the value of diagonal term Q_{aa} .

The Replica Fourier Transform \widehat{Q}_{ab} of the matrix Q_{ab}

is defined by [24,31]:

$$\widehat{Q}_k = \sum_{r=k}^{R+1} p_r (Q_r - Q_{r-1}), \quad k = 0, 1, \dots, R+1 \quad (93)$$

where we have taken $Q_{-1} = 0$. In the following all quantities with index out of the allowed range are assumed to be equal to zero. From the above definition, the useful relation

$$\widehat{Q}_k - \widehat{Q}_{k+1} = p_k (Q_k - Q_{k-1}) \quad (94)$$

allows the inversion of the RFT

$$Q_r = \sum_{k=0}^r \frac{1}{p_k} (\widehat{Q}_k - \widehat{Q}_{k+1}), \quad r = 0, 1, \dots, R+1. \quad (95)$$

It can be shown [24] that the Replica Fourier Transform has the properties of the standard Fourier Transform. In particular the RFT of the inverse matrix $(Q^{-1})_{ab}$ is the inverse of the RFT of Q_{ab} :

$$\widehat{Q^{-1}}_k = \frac{1}{\widehat{Q}_k}. \quad (96)$$

Then, by virtue of eqs. (94) and (95) we have for $r = 0, \dots, R$,

$$\begin{aligned} (Q^{-1})_r &= \sum_{k=0}^r \frac{1}{p_k} \left(\frac{1}{\widehat{Q}_k} - \frac{1}{\widehat{Q}_{k+1}} \right) \\ &= - \sum_{k=0}^r \frac{Q_k - Q_{k-1}}{\widehat{Q}_{k+1} \widehat{Q}_k} \end{aligned} \quad (97)$$

and

$$\begin{aligned} (Q^{-1})_{R+1} &= \frac{1}{\widehat{Q}_{R+1}} + (Q^{-1})_R \\ &= \frac{1}{Q_{R+1} - Q_R} - \sum_{k=0}^R \frac{Q_k - Q_{k-1}}{\widehat{Q}_{k+1} \widehat{Q}_k}. \end{aligned} \quad (98)$$

These expression remains valid also in the $n \rightarrow 0$ limit.

In the limit of $R \rightarrow \infty$ the difference $Q_r - Q_{r-1}$ with $r \leq R$ goes to zero, the sum can then be replaced by an

integral and we have, for $0 \leq x \leq x_c \equiv p_R$,

$$(Q^{-1})(x) = - \int_0^x ds \frac{\dot{Q}(s)}{\widehat{Q}(s)^2} - \frac{Q(0)}{\widehat{Q}(0)^2}, \quad (99)$$

where

$$\widehat{Q}(x) = Q(1) - Q(x_c) + \int_x^{x_c} ds s \dot{Q}(s). \quad (100)$$

and $Q(1) \equiv Q_{R+1}$. Similarly

$$\begin{aligned} (Q^{-1})(1) &= \frac{1}{Q(1) - Q(x_c)} + (Q^{-1})(x_c) \\ &= \frac{1}{Q(1) - Q(x_c)} - \int_0^{x_c} ds \frac{\dot{Q}(s)}{\widehat{Q}(s)^2} - \frac{Q(0)}{\widehat{Q}(0)^2}. \end{aligned} \quad (101)$$

B Computing $\text{Tr} \ln Q$

The quantity $\text{Tr} \ln Q$ can be computed by using the identity

$$\frac{1}{n} \text{Tr} \ln Q = \lim_{m \rightarrow 0} \frac{1}{m} \left(\frac{1}{n} \text{Tr} Q^m - 1 \right) \quad (102)$$

and the results of the previous Appendix.

The quantity Q^m for integer m is the matrix product of m matrices, i.e., a convolution in matrix indexes. As a consequence

$$\widehat{Q^m}_r = \widehat{Q}_r^m. \quad (103)$$

Then, by using eq. (95), we have

$$\frac{1}{n} \text{Tr} Q^m = (Q^m)_{R+1} = \sum_{k=0}^{R+1} \frac{1}{p_k} \left(\widehat{Q}_k^m - \widehat{Q}_{k+1}^m \right) \quad (104)$$

and hence

$$\begin{aligned} \frac{1}{n} \text{Tr} \ln Q &= \sum_{k=0}^{R+1} \frac{1}{p_k} \left(\ln \widehat{Q}_k - \ln \widehat{Q}_{k+1} \right) \\ &\stackrel{n \rightarrow 0}{=} \ln(Q_{R+1} - Q_R) + \sum_{k=1}^R \frac{1}{p_k} \ln \left(\frac{\widehat{Q}_k}{\widehat{Q}_{k+1}} \right) \\ &\quad + \frac{Q_0}{\widehat{Q}_1}. \end{aligned} \quad (105)$$

This expression is valid for any finite R . To perform the limit we use the relation (94) and expand the logarithm in the sum for small $Q_r - Q_{r-1}$, i.e.,

$$\begin{aligned} \ln\left(\frac{\hat{Q}_k}{\hat{Q}_{k+1}}\right) &= \ln\left(1 + p_k \frac{Q_k - Q_{k-1}}{\hat{Q}_{k+1}}\right) \\ &= p_k \frac{Q_k - Q_{k-1}}{\hat{Q}_k} + O((Q_k - Q_{k+1})^2) \end{aligned} \quad (106)$$

so that eq. (105) becomes

$$\begin{aligned} \frac{1}{n} \text{Tr} \ln Q &= \ln(Q_{R+1} - Q_R) + \sum_{k=1}^R \frac{Q_k - Q_{k-1}}{\hat{Q}_k} \\ &\quad + \frac{Q_0}{\hat{Q}_1} + \sum_{k=1}^R O((Q_k - Q_{k+1})^2). \end{aligned} \quad (107)$$

In the limit $R \rightarrow \infty$ the last sum vanishes and we end up with

$$\begin{aligned} \frac{1}{n} \text{Tr} \ln Q &= \ln(Q(1) - Q(x_c)) \\ &\quad + \int_0^{x_c} dx \frac{\dot{Q}(x)}{\hat{Q}(x)} + \frac{Q(0)}{\hat{Q}(0)} \end{aligned} \quad (108)$$

C Proof of eq. (47)

If we neglect the term $\mathcal{K}_2[G, A]$, and use $G_{ab} = Q_{ab}$ the functional \mathcal{L} can be written as [27,33]:

$$\begin{aligned} \frac{2}{n} \mathcal{L} &= \frac{\beta^2}{2n} \sum_{ab} Q_{ab}^2 + \frac{1}{n} \text{Tr} \ln Q + \text{const} \\ &= \beta^2 \int_0^1 dQ Q x(Q) + \int_0^{Q_R} \frac{dQ}{\int_Q^1 dQ' x(Q')} \\ &\quad + \ln(1 - Q_R) + \text{const} \end{aligned} \quad (109)$$

where

$$x(Q) = p_0 + \sum_{r=0}^R (p_{r+1} - p_r) \theta(Q - Q_r). \quad (110)$$

The saddle point equations are obtained by varying the above functional with respect to $x(Q)$:

$$\frac{2}{n} \mathcal{L} = \int_0^1 dQ F(Q) \delta x(Q) \quad (111)$$

where

$$F(Q) = \beta^2 Q - \int_0^Q \frac{dQ'}{\left[\int_{Q'}^1 dQ'' x(Q'')\right]^2} \quad (112)$$

and

$$\begin{aligned} \delta x(Q) &= \sum_{r=0}^R (\delta p_{r+1} - \delta p_r) \theta(Q - Q_r) \\ &\quad - \sum_{r=0}^R (p_{r+1} - p_r) \delta(Q - Q_r) \delta Q_r. \end{aligned} \quad (113)$$

By requiring the stationarity of \mathcal{L} with respect to variations of Q_r and p_r one gets, respectively

$$F(Q_r) = 0, \quad r = 0, \dots, R, \quad (114)$$

$$\int_{Q_{r-1}}^{Q_r} dq F(Q) = 0, \quad r = 1, \dots, R. \quad (115)$$

The function $F(Q)$ is continuous, thus Eq. (115) implies that between any two successive Q_r there must be at least two extrema of $F(Q)$. If we denote these by Q^* , then the extremal condition $F'(Q^*) = 0$ implies that

$$\int_{Q^*}^1 dq x(q) = T. \quad (116)$$

The right hand side of this equation is a constant and hence there is only one solution (or none). Thus the only solution of eq. (115) is $Q_{r-1} = Q_r = Q$ for all r . This in turn implies that $x(Q') = \theta(Q' - Q)$, so that eq. (114) reduces to

$$\beta^2 Q - \frac{Q}{(1-Q)^2} = 0 \quad (117)$$

that is, $Q = 1 - T$.

D Double Legendre Transform

In this section we give a short summary of the double Legendre Transform to help to understand the origin of

eq. (4). To keep the notation as simple as possible we shall consider the case of a single scalar field. The extension to more complicated cases is straightforward.

We are interested in the evaluation of

$$\ln Z = \ln \int d\phi e^{-S[\phi]} \quad (118)$$

for some $S[\phi]$. To this end we define the generating function $W[J, K]$ as [18,19]

$$e^{W[J, K]} = \int d\phi e^{-S[\phi] + J\phi + \frac{1}{2}\phi K\phi}. \quad (119)$$

Clearly

$$\ln Z = W[J, K] \Big|_{J=K=0}. \quad (120)$$

From the definition of $W[J, K]$ it follows

$$\frac{\partial}{\partial J} W[J, K] = \langle \phi \rangle = \varphi \quad (121)$$

$$\frac{\partial}{\partial K} W[J, K] = \frac{1}{2} \langle \phi \phi \rangle = \frac{1}{2} (\varphi \varphi + G) \quad (122)$$

where the average $\langle \cdots \rangle$ is taken with the full weight

$$\exp \left[-S[\phi] + J\phi + \frac{1}{2}\phi K\phi \right] \quad (123)$$

and G is the connected two-point correlator. We can then define the double Legendre Transform $\Gamma[\varphi, G]$ of $W[J, K]$ as

$$W[J, K] + \Gamma[\varphi, G] = J\varphi + \frac{1}{2}\varphi K\varphi + \frac{1}{2}KG \quad (124)$$

with

$$\frac{\partial}{\partial \varphi} \Gamma[\varphi, G] = J + K\varphi, \quad (125)$$

$$\frac{\partial}{\partial G} \Gamma[\varphi, G] = \frac{1}{2}K. \quad (126)$$

Comparison of eq. (120) with eqs. (124)-(126) shows that

$$\ln Z = -\Gamma[\varphi, G] \quad (127)$$

where φ and G are solution of the equations

$$\frac{\partial}{\partial \varphi} \Gamma[\varphi, G] = 0 \quad (128)$$

$$\frac{\partial}{\partial G} \Gamma[\varphi, G] = 0 \quad (129)$$

i.e., the value of $\ln Z$ is equal to (minus) the value of $\Gamma[\varphi, G]$ at its stationarity point.

Up to now we have just used some general properties of the Legendre Transform. The usefulness of this approach arises from the fact that one has an explicit form of $\Gamma[\varphi, G]$. Indeed one has

$$\begin{aligned} \Gamma[\varphi, G] = & S[\varphi] + \frac{1}{2} \text{Tr} \ln G^{-1} + \frac{1}{2} \text{Tr} D^{-1} G \\ & + \mathcal{K}_2[\varphi, G] - \frac{1}{2} \text{Tr} 1 \end{aligned} \quad (130)$$

where

$$D^{-1}[\varphi] = \frac{\partial^2}{\partial \phi \partial \phi} S[\phi] \Big|_{\phi=\varphi} \quad (131)$$

and $\mathcal{K}_2[\varphi, G]$ is given by the sum of all 2-PI vacuum diagrams of a theory with propagators G and interaction vertices determined by the potential V_{int} given by

$$S[\varphi+\phi] - S[\varphi] - \phi \frac{\partial}{\partial \phi} S[\phi] \Big|_{\phi=\varphi} = \frac{1}{2} \phi D^{-1}[\varphi]\phi + V_{\text{int}}[\phi; \varphi]. \quad (132)$$

This procedure corresponds to a dressed loop expansion with vertices that depend on φ and can thus exhibit non-perturbative effects even for a small number of dressed loops. The stationarity condition on $\Gamma[\varphi, G]$ yields a set of coupled (nonlinear) equations for φ and G .

References

1. D. Sherrington and S. Kirkpatrick, Phys. Rev. B **17**, 4384 (1978)

2. Notice that $\Omega[Q]$ is disorder independent. If we were to study another 2-spin model (ROM, etc...), the Lagrangean would read $\mathcal{L}[Q] = G[Q] + \Omega[Q]$ where $\Omega[Q]$ is as defined in eq. (5) and $G[Q]$ determined as in Ref. [3]. The Lagrangean can be extended to more complex spin interactions following Ref. [4]. In this work we keep to the SK model.
3. R. Cherrier, D.S. Dean, A. Lefevre, Phys. Rev E **67**, 046112 (2003); T. Sarlat, Thesis, 2009.
4. A. Crisanti, G. Parisi and L. Leuzzi, J. Phys. A **35**, 481 (2002)
5. G. Parisi, Phys. Rev. Lett. **43**, 1754 (1979)
6. G. Parisi, J. Phys. A **13**, 1101 (1980)
7. B. Duplantier, J. Phys. A. **14**, 283 (1981)
8. G. Parisi, Phys. Rev. Lett. **50**, 1946 (1983)
9. A. Crisanti and T. Rizzo, Phys. Rev. E **65**, 46137 (2002)
10. A. Crisanti, T. Rizzo and T. Temesvari, Eur. Phys. J. B **33**, 203 (2003)
11. S. Pankov, Phys. Rev. Lett. **96**, 197204 (2006)
12. R. Oppermann and D. Sherrington, Phys. Rev. Lett. **95**, 197203 (2005)
13. R. Oppermann, M.J. Schmidt and D. Sherrington, Phys. Rev. Lett. **98**, 127201 (2007)
14. M.J. Schmidt and R. Oppermann, Phys. Rev. E **77**, 061104 (2008)
15. M.J. Schmidt, Dissertation, Würzburg 2008, (unpublished)
16. R. Oppermann and M.J. Schmidt, Phys. Rev. E **78**, 061124 (2008)
17. C. De Dominicis and P.C. Martin, J. Math. Phys. **5**, 14 and 31 (1964)
18. J.M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. **10**, 2428 (1974)
19. R.W. Haymaker, Riv. Nuovo Cimento **14**, 1 (1991)
20. If one keeps the $\langle \sigma_a \lambda_b \rangle$ correlators then their equation of motion only admits the trivial null solution.
21. We leave aside the possibility of having vectorial replica symmetry breaking.
22. For instance in the $q \geq 4$ Potts model the symmetry is broken and indeed $Q_0 \neq 0$.
23. T Temesvari, C. De Dominicis and I. Kondor, J. Phys. A **27**, 7569 (1994)
24. C. De Dominicis, D.M. Carlucci and T. Temesvari, J. Phys. I France **7**, 105 (1997)
25. If all 2-PI diagrams coming from K_2 are omitted the model reduces to the 2-spin spherical model; J. M. Kosterlitz, D. J. Thouless, and R. C. Jones Phys. Rev. Lett. **36** 1217 (1976)
26. T.R. Kirkpatrick and D. Thirumalai, Phys. Rev. B **36**, 5388 (1987)
27. A. Crisanti and H.-J. Sommers, Z. Phys. B **87**, 341 (1992).
28. A. Crisanti, H. Horner, and H.J. Sommers, Z. für Phys. B **92**, 257 (1993).
29. R. Monasson, Phys. Rev. Lett. **75**, 2847 (1995)
30. C. De Dominicis, T. Temesvari and I. Kondor, J. de Physique IV France **8**, 13 (1998) (Preprint cond-mat/9802166) Equation numbering having been messed up at the editing stage, the reader should rather consult the cond-mat version.
31. The RFT was first introduced, directly in the continuum limit ($R \rightarrow \infty$) by Mezard and Parisi[32]
32. M. Mezard and G. Parisi, J. Phys. I France **1**, 809 (1991).
33. A. Crisanti and L. Leuzzi, Phys. Rev. B **73**, 014412 (2006).

